Electric Polarizability and Magnetic Susceptibility of Small Holes in a Thin Screen

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Abstract —**Frequently in the generation and transmission of RF waves, different regions of excitation are coupled by a small aperture in a common plane wall. When the dimensions of the aperture are small compared to the wavelength, the effect of the aperture can be described by an electric polarizability, χ, and magnetic susceptibilities ψ_{xx} and ψ_{yy}, which are defined in static terms. Specific results for χ, ψ_{xx}, and ψ_{yy} have been derived by Bethe [1] and Collin [2] for circular and elliptical holes. We have derived integral equations for the field components in the plane of the hole and variational forms for χ, ψ_{xx}, and ψ_{yy} in terms of these field components. We have also shown that the polarizability and (diagonalized) susceptibility are connected by $1/\chi = 1/\psi_{xx} + 1/\psi_{yy}$, a relation which does not appear in any of the related literature which we have examined for an aperture of general shape.**

I. Introduction

IN THE DESIGN of RF structures for use in the generation and propagation of microwaves there are many applications where two or more regions are coupled through a hole in a thin metallic screen. When the largest dimension of the hole is small compared to all other significant lengths, such as the RF wavelength, the radius of curvature of the wall at the hole, or the distance to the nearest important discontinuity, it is well known (see, for example [1]–[3]) that the electromagnetic properties of the hole can be represented by an induced electric moment, perpendicular to the plane of the hole, and by an induced (vector) magnetic moment, in the plane of the hole.

In this paper we develop methods to solve the electrostatic and magnetostatic problems in order to obtain results for χ, the electric polarizability, and $\overleftrightarrow{\psi}$, the magnetic susceptibility, of a hole of general shape in a plane metallic screen. In the process, we discover that χ and $\overleftrightarrow{\psi}$ are simply related to one another, a fact that does not seem to have been noted previously.

II. Coupling Integral

Let us consider a cavity of general shape whose boundary contains a small hole, and expand the fields in terms of the orthonormal complete set [4] of field functions in the absence of the hole.[1] These modes are solutions of the equations

$$\nabla \times \vec{e}_m(\vec{x}) = k_m \vec{h}_m(\vec{x}) \qquad \nabla \times \vec{h}_m(\vec{x}) = k_m \vec{e}_m(\vec{x}) \tag{1}$$

and satisfy the orthonormality condition

$$\int dv\, \vec{e}_m \cdot \vec{e}_{m'} = \int dv\, \vec{h}_m \cdot \vec{h}_{m'} = \delta_{mm'} \tag{2}$$

where $k_m c/2\pi$ are the eigenfrequencies of the cavity. In addition, $\vec{e}_m$ and $\vec{h}_m$ satisfy the usual boundary condition at a metal surface:

$$\vec{e}_m \times \vec{n} = 0 \qquad \vec{h}_m \cdot \vec{n} = 0. \tag{3}$$

The actual steady-state fields $\vec{E}(\vec{x})e^{j\omega t}$ and $\vec{H}(\vec{x})e^{j\omega t}$ in the presence of the hole satisfy Maxwell's equations

$$\nabla \times \vec{E}(\vec{x}) = -j\omega\mu \vec{H}(\vec{x}) \tag{4}$$

$$\nabla \times \vec{H}(\vec{x}) = j\omega\epsilon \vec{E}(\vec{x}) \tag{5}$$

where ϵ and μ are the permittivity and permeability of the cavity medium.

It is possible to show that the fields in the interior of the cavity can be expressed as an integral over the electric field in the plane of the hole (see, for example, [3] and [4]). Specifically, we can write

$$\vec{E}(\vec{x}) = \sum_m \vec{e}_m(\vec{x}) \frac{k_m J_m}{k^2 - k_m^2} \tag{6}$$

$$\vec{H}(\vec{x}) = j\omega\epsilon \sum_m \vec{h}_m(\vec{x}) \frac{J_m}{k^2 - k_m^2} \tag{7}$$

where $k = \omega/c$, corresponding to the vacuum values of ϵ and μ, and where the coupling integral over the area of the hole is given by

$$J_m = \int dS \left(\vec{n} \cdot \vec{E} \times \vec{h}_m\right). \tag{8}$$

Some comments are necessary at this point. There are convergence problems if (6) and (7) are used in the vicinity of the hole. For example, if (6) is used in the plane of the

[1] It should be noted that the set of functions that we are using for the electric field is complete only in the absence of charge.

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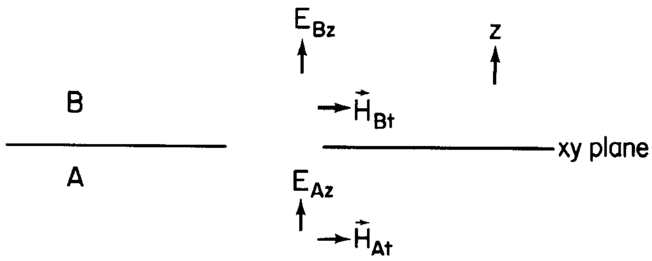

Fig. 1. Electrostatic and magnetostatic fields near the hole.

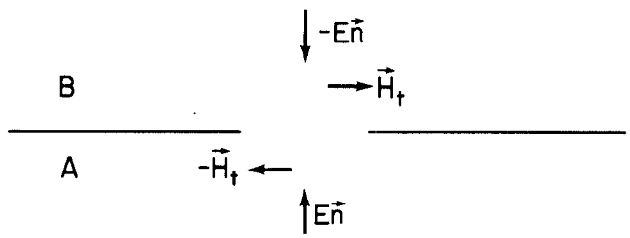

Fig. 2. Antisymmetrized electric and magnetic fields near the hole.

hole, $\vec{E}(\vec{x})$ will (incorrectly) be normal to the hole surface since $\vec{e}_m \times \vec{n} = 0$ within the hole. Because of this, the way in which we shall proceed is to evaluate the normal component of $\vec{E}(\vec{x})$ in the vicinity of the hole using (6) and subsequently use this normal component as the "far field" in treating the "electrostatic" problem in the immediate vicinity of the hole. Similar considerations apply to (7), to the tangential magnetic field in the vicinity of the hole, and to the subsequent "magnetostatic" analysis in the immediate vicinity of the hole. Clearly this procedure is valid only if the hole dimensions are small compared to the wavelength and all other significant dimensions. We will therefore write, for the normal electric field and the tangential magnetic field in the vicinity of the hole,

$$E_{Az}(0) = \sum_m e_{mz}(0) \frac{k_m J_m}{k^2 - k_m^2}$$

$$Z_0 \vec{H}_{At}(0) = jk \sum_m \vec{h}_m(0) \frac{J_m}{k^2 - k_m^2} \tag{9}$$

where the subscript A refers to cavity A, (0) stands for the coordinates of the hole center, z is perpendicular to the plane of the hole, t means parallel to the plane of the hole, and $Z_0 = \sqrt{\mu/\epsilon} = 120\pi\ \Omega$ is the impedance of free space.

We now consider two general cavities, A and B, coupled by the same hole. The above analysis applies as well in cavity B and we obtain

$$E_{Bz}(0) = \sum_l e_{lz}(0) \frac{k_l J_l}{k^2 - k_l^2}$$

$$Z_0 \vec{H}_{Bt}(0) = jk \sum_l \vec{h}_{lt}(0) \frac{J_l}{k^2 - k_l^2}. \tag{10}$$

Here the summation index m applies to cavity A, the summation index l applies to cavity B, and

$$J_l = -\int ds \vec{n} \cdot \vec{E} \times \vec{h}_l \tag{11}$$

where $-\vec{n}$ is the outward normal to cavity B. In Fig. 1 we show the situation in the immediate vicinity of the hole, where the xy plane can be considered as extending to $\pm\infty$. We have dropped the reference to the hole coordinates (0). We must now determine the x and y components of $\vec{E}(\vec{x})$ for $z = 0$ to be used in Eqs. (8) and (11). These will clearly not change if we antisymmetrize the field configuration by subtracting the fields

$$\frac{E_{Bz} + E_{Az}}{2} \qquad \frac{\vec{H}_{Bt} + \vec{H}_{At}}{2}$$

in all space, since neither upset the boundary conditions on the metal screen; nor will the desired x and y components of $\vec{E}$ or z component of $\vec{H}$ be changed by the subtraction. The resulting antisymmetrized field configuration is shown in Fig. 2, where

$$E = \frac{E_{Az} - E_{Bz}}{2} \qquad \vec{H}_t = \frac{\vec{H}_{Bt} - \vec{H}_{At}}{2}.$$

As we shall see, the values of J_m and J_l, which have not been changed by the subtraction of the constant fields, are necessarily proportional to E and $\vec{H}_t$.

III. Evaluation of the Coupling Integral for a Small Hole

Let us consider the integral

$$J_m = \int dS \vec{n} \cdot \vec{E}(\vec{x}) \times \vec{h}_m(\vec{x}) \tag{12}$$

and expand $\vec{h}_m(\vec{x})$ in a Taylor series in $\vec{x}$ to obtain

$$J_m = J_m^0 + J_m^1 \tag{13}$$

where

$$J_m^0 = h_{my}(0) \iint dx\, dy\, E_x(x, y) - h_{mx}(0) \iint dx\, dy\, E_y(x, y) \tag{14}$$

$$\begin{aligned} J_m^1 = {} & \frac{\partial h_{my}}{\partial x}(0) \iint dx\, dy\, x E_x(x, y) \\ & + \frac{\partial h_{my}}{\partial y}(0) \iint dx\, dy\, y E_x(x, y) \\ & - \frac{\partial h_{mx}}{\partial x}(0) \iint dx\, dy\, x E_y(x, y) \\ & - \frac{\partial h_{mx}}{\partial y}(0) \iint dx\, dy\, y E_y(x, y). \end{aligned} \tag{15}$$

Here $x = y = 0$ is an arbitrary "center" of the hole. In the Appendix we show that J_m^1 corresponds to an electric dipole moment and J_m^0 to a magnetic dipole moment. Specifically, we obtain

$$J_m = k_m \chi e_{mz}(0) E - j\omega\mu \vec{h}_{mt}(0) \cdot \overleftrightarrow{\psi} \cdot \vec{H}_t \tag{16}$$

where χ and $\overleftrightarrow{\psi}$ are the electric polarizability and magnetic susceptibility of the hole. The quantity $\overleftrightarrow{\psi}$ is treated as a tensor because $\vec{H}_t$ is not necessarily parallel to the symmetry axes of the hole. We shall later show that the x and y

axes can be oriented so that $\overleftrightarrow{\psi}$ is diagonal, with components ψ_{xx} and ψ_{yy} in the x and y directions respectively.

The definition of χ given in the Appendix is

$$\chi = \frac{1}{E} \iint dx\, dy\, \Phi(x, y) \tag{17}$$

where $\Phi(x, y)$ is the solution of the electrostatic problem with constant "far fields" $\pm E\vec{n}$ as shown in Fig. 2. Similarly, the magnetic susceptibility, $\overleftrightarrow{\psi}$, is defined by

$$\psi_{xx} H_x + \psi_{xy} H_y = \iint x\, dx\, dy\, H_z(x, y) \tag{18}$$

$$\psi_{yx} H_x + \psi_{yy} H_y = \iint y\, dx\, dy\, H_z(x, y) \tag{19}$$

where $H_z(x, y)$ is the solution of the magnetostatic problem with constant "far fields" $\pm \vec{H}_t$.

IV. Magnetic Susceptibility

A. Integral Equation

For reasons that will become clear later, we will first derive equations for the magnetic susceptibility. We will first analyze the problem where $H_x = 1$, $H_y = 0$, in which case we can write the scalar magnetic potential, which must be an odd function of z, as

$$\Psi(x, y, z) = \pm \left(x - \iint dk\, dl\, b(k, l)\, e^{ikx + ily - \sqrt{k^2 + l^2}|z|} \right) \tag{20}$$

where the $\pm$ corresponds to $z \gtrless 0$, and where $\vec{H} = \nabla \Psi$. The z component of H_z at $z = 0$ is

$$H_z(x, y, 0) \equiv g(\vec{r}) = \int d\vec{\sigma}\, e^{i\vec{\sigma} \cdot \vec{r}} \sigma b(\vec{\sigma}) \tag{21}$$

where

$$\vec{\sigma} = \vec{i}k + \vec{j}l \qquad \vec{r} = \vec{i}x + \vec{j}y. \tag{22}$$

Since $H_z = 0$ on the metal, $g(x, y)$ vanishes everywhere except in the hole, and (21) can be inverted to yield

$$\sigma b(\vec{\sigma}) = \frac{1}{4\pi^2} \int_H d\vec{r}\, g(\vec{r})\, e^{-i\vec{\sigma} \cdot \vec{r}}. \tag{23}$$

The continuity of $\Psi(x, y, z)$ (which is odd in z) at $z = 0$ requires that

$$\int d\vec{\sigma}\, e^{i\vec{\sigma} \cdot \vec{r}} b(\vec{\sigma}) = x \tag{24}$$

be satisfied within the hole. Combining (23) and (24) leads to the integral equation

$$\int_H d\vec{r}'\, g(\vec{r}')\, K(\vec{r}, \vec{r}') = x \tag{25}$$

where the kernel

$$K(\vec{r}, \vec{r}') = \frac{1}{4\pi^2} \int \frac{d\vec{\sigma}}{\sigma} e^{i\vec{\sigma} \cdot (\vec{r} - \vec{r}')} = \frac{1}{2\pi} \int_0^\infty d\sigma\, J_0(\sigma |\vec{r} - \vec{r}'|) = \frac{1}{2\pi |\vec{r} - \vec{r}'|} \tag{26}$$

is symmetric in $\vec{r}$ and $\vec{r}'$, and where we have used the value of the integral of the zero-order Bessel function:

$$\int_0^\infty dt\, J_0(t) = 1. \tag{27}$$

Before proceeding further, let us treat the problem where $H_x^H = 0$, $H_y^H = 1$. A parallel analysis for

$$H_z(x, y, 0) \equiv h(\vec{r}) \tag{28}$$

leads to the integral equation

$$\int d\vec{r}'\, h(\vec{r}')\, K(\vec{r}, \vec{r}') = y \tag{29}$$

where $K(\vec{r}, \vec{r}')$ is the same as that in (26). We now have

$$\psi_{xx} = \int d\vec{r}\, x g(\vec{r}) \qquad \psi_{yx} = \int d\vec{r}\, y g(\vec{r})$$
$$\psi_{xy} = \int d\vec{r}\, x h(\vec{r}) \qquad \psi_{yy} = \int d\vec{r}\, y h(\vec{r}). \tag{30}$$

B. Diagonalization of ψ_{ij}

If we multiply (25) by $h(\vec{r})$ and integrate over $\vec{r}$, we obtain

$$\psi_{xy} = \int d\vec{r} \int d\vec{r}'\, h(\vec{r})\, g(\vec{r}')\, K(\vec{r}, \vec{r}'). \tag{31}$$

In a similar way,

$$\psi_{yx} = \int d\vec{r} \int d\vec{r}'\, g(\vec{r})\, h(\vec{r}')\, K(\vec{r}, \vec{r}'). \tag{32}$$

Since $K(\vec{r}, \vec{r}')$ is symmetric in $\vec{r}$ and $\vec{r}'$ we have

$$\psi_{xy} = \psi_{yx} \tag{33}$$

and the 2×2 matrix in (30) can be diagonalized; that is, a set of orthogonal axes x', y' can be found in which a field along the x' or y' axis induces a magnetic moment along the same axis. We now assume that x and y have been chosen to be these "diagonalized" axes and have for the susceptibilities

$$\psi_{xx} = \int d\vec{r}\, x g(\vec{r}) \qquad \psi_{yy} = \int d\vec{r}\, y h(\vec{r}) \qquad \psi_{xy} = \psi_{yx} = 0. \tag{34}$$

V. Electric Polarizability

For an asymptotic field $\pm E\vec{n}$ as shown in Fig. 2 we can write the electrostatic potential, which is an even function of z, for $E = 1$ as

$$\Phi(x, y, z) = |z| + \int d\vec{\sigma}\, e^{i\vec{\sigma} \cdot \vec{r} - \sigma |z|} a(\vec{\sigma}). \tag{35}$$

Since the potential vanishes on the metal surface we have, over the surface of the hole,

$$\Phi(x,y,0) \equiv f(\vec{r}) = \int d\vec{\sigma} e^{i\vec{\sigma}\cdot\vec{r}} a(\vec{\sigma}) \tag{36}$$

and its inverse

$$a(\vec{\sigma}) = \frac{1}{4\pi^2}\int_H d\vec{r} e^{-i\vec{\sigma}\cdot\vec{r}} f(\vec{r}). \tag{37}$$

Continuity of $E_z(x,y,0)$ within the hole requires that

$$\int d\vec{\sigma} e^{i\vec{\sigma}\cdot\vec{r}} \sigma a(\vec{\sigma}) = 1 \tag{38}$$

which is equivalent to $E_z(x,y,0)=0$. If we proceed as before to substitute (37) into (38), we obtain the integral equation

$$\int d\vec{r}' f(\vec{r}')\hat{K}(\vec{r},\vec{r}') = 1 \tag{39}$$

where

$$\hat{K}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int d\vec{\sigma} \sigma e^{i\vec{\sigma}\cdot(\vec{r}-\vec{r}')}. \tag{40}$$

It is not hard to show that

$$\hat{K}(\vec{r},\vec{r}') = -\nabla^2_{\vec{r}} K(\vec{r},\vec{r}') = -\nabla^2_{\vec{r}'} K(\vec{r},\vec{r}') \tag{41}$$

with $K(\vec{r},\vec{r}')$ given in (26), is highly singular at $\vec{r}=\vec{r}'$. This mathematical difficulty has undoubtedly come about because of the interchange of order of integration between $\vec{r}'$ and $\vec{\sigma}$ in obtaining (39). As an attempt to repair the problem, we rewrite (40) as

$$\int d\vec{r}' f(\vec{r}')\nabla_{\vec{r}}\cdot\nabla_{\vec{r}'} K(\vec{r},\vec{r}') = 1 \tag{42}$$

which would be satisfied by

$$\int d\vec{r}' f(\vec{r}')\nabla_{r'}\vec{K}(\vec{r},\vec{r}') = \vec{i}\alpha x + \vec{j}\beta y \tag{43}$$

where α and β are to be determined later, subject to the condition

$$\alpha + \beta = 1. \tag{44}$$

If we now integrate the left side of (43) by parts and take $f=0$ along the boundary of the hole, we find

$$-\int d\vec{r}' \frac{\partial f}{dx'} K(\vec{r},\vec{r}') = \alpha x \tag{45}$$

$$-\int d\vec{r}' \frac{\partial f}{\partial y'} K(\vec{r},\vec{r}') = \beta y \tag{46}$$

in which there is no longer a problem with the singularity at $\vec{r}=\vec{r}'$. Equations (45) and (46) are now identical to (25) and (29), from which we conclude that

$$\frac{\partial f(x,y)}{\partial x} = -\alpha g(x,y) \tag{47}$$

$$\frac{\partial f(x,y)}{\partial y} = -\beta h(x,y). \tag{48}$$

If we now multiply (47) by x, (48) by y, and integrate by parts on the left side, we find

$$\chi = \iint dx\,dy f(x,y) = \alpha \iint dx\,dy\,x g(x,y) = \beta \iint dx\,dy\,y h(x,y) \tag{49}$$

from which we can write

$$\alpha\psi_{xx} = \beta\psi_{yy} = \chi. \tag{50}$$

Equations (44) and (50) lead directly to

$$\alpha = \frac{\psi_{yy}}{\psi_{xx}+\psi_{yy}} \qquad \beta = \frac{\psi_{xx}}{\psi_{xx}+\psi_{yy}} \tag{51}$$

from which one obtains the general relation between the electric and magnetic susceptibilities

$$\frac{1}{\chi} = \frac{1}{\psi_{xx}} + \frac{1}{\psi_{yy}}. \tag{52}$$

As mentioned earlier, ψ_{xx} and ψ_{yy} are the components of the diagonalized susceptibility.

Exact expressions have been obtained for the polarizability and susceptibilities of an elliptical hole (see, for example [2]). These can be written as

$$\frac{1}{\chi} = \frac{3}{8\pi ab}\int_0^{2\pi} d\psi \left(\frac{\cos^2\psi}{a^2} + \frac{\sin^2\psi}{b^2}\right)^{1/2} \tag{53}$$

$$\frac{1}{\psi_{xx}} = \frac{3}{8\pi ab}\int_0^{2\pi} d\psi \frac{\dfrac{\cos^2\psi}{a^2}}{\left(\dfrac{\cos^2\psi}{a^2} + \dfrac{\sin^2\psi}{b^2}\right)^{1/2}} \tag{54}$$

$$\frac{1}{\psi_{yy}} = \frac{3}{8\pi ab}\int_0^{2\pi} d\psi \frac{\dfrac{\sin^2\psi}{b^2}}{\left(\dfrac{\cos^2\psi}{a^2} + \dfrac{\sin^2\psi}{b^2}\right)^{1/2}} \tag{55}$$

where the integrals can readily be expressed in terms of complete elliptic integrals of the first and second kinds. Here a and b are the semimajor and semiminor axes of the elliptical hole. The validity of (52) in this case is obvious. Corresponding expressions can easily be obtained for a circular hole, where $\psi_{xx}=\psi_{yy}=2\chi$.

To complete this section we also give variational forms for ψ_{xx} and ψ_{yy} which can be used to obtain reasonably accurate susceptibilities with only approximate values of

$g(\vec{r})$ and $h(\vec{r})$. Specifically, we have

$$\frac{1}{\psi_{xx}} = \frac{\int d\vec{r} \int d\vec{r}' g(\vec{r}) g(\vec{r}') K(\vec{r}, \vec{r}')}{\left[\int d\vec{r} x g(\vec{r})\right]^2} \tag{56}$$

$$\frac{1}{\psi_{yy}} = \frac{\int d\vec{r} \int d\vec{r}' h(\vec{r}) h(\vec{r}') K(\vec{r}, \vec{r}')}{\left[\int d\vec{r} y h(\vec{r})\right]^2} \tag{57}$$

where $g(\vec{r}) = g(x, y)$ and $h(\vec{r}) = h(x, y)$ are trial functions and where $K(\vec{r}, \vec{r}')$ is defined in (26). The equivalent evaluation of the polarizability is obtained by using (52) with the variational values of ψ_{xx}^{-1} and ψ_{yy}^{-1} obtained from (56) and (57). If the boundary curve in the x, y plane is defined by

$$p(x, y) = p_0 \tag{58}$$

then trial functions satisfying the proper edge conditions can be taken to be

$$g(x, y) = \frac{\partial}{\partial x}(p_0 - p(x, y))^{1/2} \tag{59}$$

$$h(x, y) = \frac{\partial}{\partial y}(p_0 - p(x, y))^{1/2}. \tag{60}$$

VI. Summary

In Section II we showed that the fields inside a cavity with a hole can be written in terms of a coupling integral involving the tangential electric field in the plane of the hole. For a hole whose dimensions are small compared to the wavelength, this coupling integral was separated in Section III into an electric term, proportional to the (scalar) polarizability of the hole, and a magnetic term, proportional to the (vector) susceptibility of the hole.

In Sections IV and V we developed integral equations for the determination of these "static" geometrical parameters. These equations can be solved exactly for circular and elliptical shape (see, for example, [1] and [2]) but only approximately for other shapes (see, for example [5]). In separate work [6] we discuss in detail variational forms for these parameters which can be used to obtain reasonably accurate values of these parameters with only very approximate trial functions.

Our main result is the derivation of what appears to be a new relation between the electric polarizability and the (diagonalized) magnetic susceptibility for a hole of general shape, namely

$$\frac{1}{\chi} = \frac{1}{\psi_{xx}} + \frac{1}{\psi_{yy}}. \tag{61}$$

This relationship can also be stated as

$$\chi^{-1} = \mathrm{Tr}\left(\overleftrightarrow{\psi}^{-1}\right) \tag{62}$$

prior to diagonalization of $\overleftrightarrow{\psi}$.

The validity of (61), which is readily confirmed for the elliptical hole, was noted in that case by Kleinman and Senior [7]. They also quote approximate numerical results for other shapes [8]–[10], which do not quite satisfy (61). We believe that the derivation of (61) is correct for any shape hole and therefore believe that the computed results are somewhat less accurate than previously estimated.

Appendix

In Section III, we separated the coupling integral of (8) into two terms, given in (14) and (15) and reproduced here:

$$J_m^0 = h_{my}(0) \iint dx\, dy\, E_x(x, y) - h_{mx}(0) \iint dx\, dy\, E_y(x, y) \tag{A1}$$

$$\begin{aligned} J_m^1 = {} & \frac{\partial h_{my}}{\partial x}(0) \iint dx\, dy\, x E_x(x, y) \\ & + \frac{\partial h_{my}}{\partial y}(0) \iint dx\, dy\, y E_x(x, y) \\ & - \frac{\partial h_{mx}}{\partial x}(0) \iint dx\, dy\, x E_y(x, y) \\ & - \frac{\partial h_{mx}}{\partial y}(0) \iint dx\, dy\, y E_y(x, y). \end{aligned} \tag{A2}$$

Here $x = y = 0$ is an arbitrary "center" of the hole. If we use the electrostatic approximation

$$\vec{E}(\vec{x}) = -\nabla \Phi(\vec{x}) \tag{A3}$$

it is easy to show that

$$\iint dx\, dy\, E_x(x, y) = -\oint dy\, \Phi(\text{boundary}) = 0 \tag{A4}$$

since the electrostatic potential vanishes on the boundary. Similarly

$$\iint dx\, dy\, E_y(x, y) = 0 \tag{A5}$$

$$\iint dx\, dy\, y E_x(x, y) = \iint dx\, dy\, x E_y(x, y) = 0 \tag{A6}$$

and, by an integration by parts,

$$\begin{aligned} \iint dx\, dy\, x E_x(x, y) &= \iint dx\, dy\, y E_y(x, y) \\ &= \iint dx\, dy\, \Phi(x, y). \end{aligned} \tag{A7}$$

We therefore can write, using (1),

$$\begin{aligned} J_m^1 &= \left(\frac{\partial h_{my}}{\partial x} - \frac{\partial h_{mx}}{\partial y}\right)_0 \iint dx\, dy\, \Phi(x, y) \\ &= k_m e_{mz}(0) \iint dx\, dy\, \Phi(x, y). \end{aligned} \tag{A8}$$

The electrostatic approximation to J_m^0 vanishes so that we must consider the time-dependent behavior of $\vec{E}$ to obtain a suitable expression for J_m^0. If we start with the integrals

$$M_x = \iint dx\,dy\,x H_z(x, y) \qquad M_y = \iint dx\,dy\,y H_z(x, y) \tag{A9}$$

and use (4), we find

$$M_x = \frac{-1}{j\omega\mu} \iint dx\,dy\,x \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \tag{A10}$$

$$M_y = \frac{1}{j\omega\mu} \iint dx\,dy\,y \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right). \tag{A11}$$

Integration by parts leads to

$$M_x = \frac{1}{j\omega\mu} \iint dx\,dy\,E_y \qquad M_y = -\frac{1}{j\omega\mu} \iint dx\,dy\,E_x \tag{A12}$$

where we have used the fact that

$$(E_x\,dx + E_y\,dy) = 0 \tag{A13}$$

on the boundary of the hole. Thus we find from (A1) that

$$J_m^0 = -j\omega\mu\left[h_{mx}(0) M_x + h_{my}(0) M_y \right]$$
$$= -j\omega\mu \iint dx\,dy\,H_z(x, y)\left[x h_{mx}(0) + y h_{my}(0) \right]. \tag{A14}$$

It is clear from the symmetry in Fig. 2 that $H_z(x, y)$ is the only nonvanishing component of $\vec{H}$ in the hole for $z = 0$. We then see from (A8) and (A14) that J_m^0 will be proportional to $\vec{H}^H$ and that J_m^1 will be proportional to $\vec{E}^H$. We therefore write for, $J_m = J_m^0 + J_m^1$,

$$J_m = k_m \chi \vec{e}_m^{\,H} \cdot \vec{E}^H - j\omega\mu \vec{h}_m^H \cdot \overleftrightarrow{\psi} \cdot \vec{H}^H \tag{A15}$$

where χ and $\overleftrightarrow{\psi}$ are the electric polarizability and magnetic susceptibility of the hole. The quantity $\overleftrightarrow{\psi}$ is treated as a tensor because $\vec{H}^H$ is not necessarily parallel to the symmetry directions of the hole. It was shown that we can orient the x and y axes so that $\overleftrightarrow{\psi}$ is diagonal, with components ψ_{xx} and ψ_{yy} in the x and y directions.

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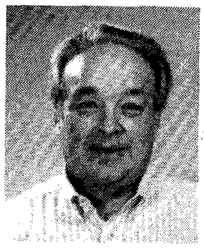

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